

Relativistic Acoustic Geometry

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Abstract

Sound wave propagation in a relativistic perfect fluid with a non-homogeneous isentropic flow is studied in terms of acoustic geometry. The sound wave equation turns out to be equivalent to the equation of motion for a massless scalar field propagating in a curved space-time geometry. The geometry is described by the acoustic metric tensor that depends locally on the equation of state and the four-velocity of the fluid. For a relativistic supersonic flow in curved space-time the ergosphere and acoustic horizon may be defined in a way analogous to the non-relativistic case. A general-relativistic expression for the acoustic analog of surface gravity has been found.

1 Introduction

Since Unruh's discovery [1] that a supersonic flow may cause Hawking radiation, the analogy between the theory of supersonic flow and the black hole physics has been extensively discussed [2, 3, 4, 5, 6, 7]. All the works have considered non-relativistic flows in flat space-time. In a comprehensive review [7] Visser defined the notions of ergo-region, acoustic apparent horizon and acoustic event horizon, and investigated the properties of various acoustic geometries.

In this paper we study the acoustic geometry for relativistic fluids moving in curved space-time. Under extreme conditions of very high density and temperature, the velocity of the fluid may be comparable with the speed of light c . In such conditions, the speed of sound may also be close to c . In particular, the equation of state of an ideal ultrarelativistic gas yields the speed of sound as $c_s = c/\sqrt{3}$. Such conditions are physically realistic in astrophysics, early big-bang cosmology and relativistic collisions of elementary particles and ions.

We restrict attention to perfect, isentropic fluids and, in addition, we impose that the flow be irrotational. In most of our considerations we make no assumption about the space-time metric. However, in the calculation of surface gravity we assume stationary space-time and steady flow. Therefore, in our discussion we make no distinction between acoustic apparent and event horizons.

We organize the paper as follows. In sections 2 and 3 we derive the relativistic acoustic wave equation in terms of an acoustic metric tensor. Section 4 describes the notions of acoustic horizon and ergo-region. In section 5 we give the calculation of surface gravity at the horizon and also study the acoustic Unruh effect and Hawking temperature. We draw our conclusions in section 6. In appendix A we outline the basic relativistic fluid dynamics and derive a few formulae needed for our consideration.

2 Relativistic Acoustics

We begin by deriving the sound wave equation for a perfect, irrotational gravitating fluid. We denote by u_μ , p , ρ , n and s the velocity, pressure, energy density, particle number density and entropy density of the fluid. The basic equations of relativistic fluid dynamics (for details see appendix A) needed for our discussion are the continuity equation

$$\partial_\mu(\sqrt{-g} n u^\mu) = 0 \quad (1)$$

and the equation of potential flow

$$w u_\mu = -\partial_\mu \phi, \quad (2)$$

where the specific enthalpy w is defined as

$$w = \frac{p + \rho}{n}. \quad (3)$$

Following the standard procedure [7, 8] we linearize equations (1) and (2) by introducing

$$\begin{aligned} w &\rightarrow w + \delta w, & n &\rightarrow n + \delta n, \\ u^\mu &\rightarrow u^\mu + \delta u^\mu, & \phi &\rightarrow \phi + \delta \phi, \end{aligned} \quad (4)$$

where the quantities δw , δn , δu^μ and $\delta \phi$ are small acoustic disturbances around some average bulk motion represented by w , n , u^μ and ϕ . For notational simplicity, in our equations we shall use φ instead of $\delta \phi$. Fluctuations of the metric due to acoustic disturbances will be neglected. Note that the normalization (72) implies

$$g_{\mu\nu} u^\mu \delta u^\nu = 0. \quad (5)$$

Substituting (4) into (2) and retaining only the terms of the same order of smallness, we find

$$\delta w = -u^\mu \partial_\mu \varphi \quad (6)$$

and

$$w \delta u^\mu = -g^{\mu\nu} \partial_\nu \varphi + u^\mu u^\nu \partial_\nu \varphi. \quad (7)$$

Similarly, linearization of the continuity equation (1) gives

$$\partial_\mu(\sqrt{-g} \delta n u^\mu) + \partial_\mu(\sqrt{-g} n \delta u^\mu) = 0. \quad (8)$$

Replacing δn by

$$\delta n = \left(\frac{\partial w}{\partial n}\right)^{-1} \delta w, \quad (9)$$

and substituting (6) and (7) into (8), we obtain the wave equation

$$\partial_\mu \left\{ \frac{n}{w} \sqrt{-g} \left[g^{\mu\nu} - \left(1 - \left(\frac{n}{w} \frac{\partial w}{\partial n} \right)^{-1} \right) u^\mu u^\nu \right] \right\} \partial_\nu \varphi = 0. \quad (10)$$

This equation describes the propagation of the linearized field φ . The space-time dependence of the background quantities n , w and $u_\mu = -w^{-1} \partial_\mu \phi$ is constrained by the equations of motion for a perfect, isentropic and irrotational gravitating fluid.

In the simplest case of a homogeneous flow in flat space-time, equation (10) becomes the free wave equation

$$(\partial_t^2 - c_s^2 \Delta) \varphi = 0, \quad (11)$$

where the quantity c_s is the sound wave velocity given by

$$c_s^2 = \frac{n}{w} \frac{\partial w}{\partial n} \Big|_{s/n}. \quad (12)$$

The subscript s/n denotes that the derivative is taken while keeping the specific entropy s/n constant, i.e. for an isentropic process. By making use of equation (80) from appendix A, it may be easily verified that this definition is equivalent to the standard definition of the speed of sound [8]

$$c_s^2 = \frac{\partial p}{\partial \rho} \Big|_{s/n}. \quad (13)$$

Introducing the symmetric tensor

$$f^{\mu\nu} = \frac{n}{w} \sqrt{-g} \left[g^{\mu\nu} - \left(1 - c_s^{-2} \right) u^\mu u^\nu \right], \quad (14)$$

we rewrite the wave equation (10) as

$$\partial_\mu f^{\mu\nu} \partial_\nu \varphi = 0. \quad (15)$$

This form of the acoustic wave equation will be used to construct the acoustic metric.

At this stage it is instructive to calculate the tensor $f_{\mu\nu}$ in the flat-metric non-relativistic limit. To do this, we reinstate the speed of light c and make the replacements

$$\begin{aligned} w &\rightarrow mc^2, & n &\rightarrow \rho_{\text{NR}}/m, \\ g^{\mu\nu} &\rightarrow \eta^{\mu\nu}, & u^\mu &\rightarrow (1, v^i/c), \\ c_s &\rightarrow v_s/c \end{aligned} \quad (16)$$

in (14). Retaining only the leading terms in $1/c$, we find

$$f^{\mu\nu} \rightarrow f_{\text{NR}}^{\mu\nu} = \frac{\rho_{\text{NR}}}{m^2 c^2 v_s^2} \begin{pmatrix} c^2 & cv^j \\ cv^i & -\delta^{ij} v_s^2 + v^i v^j \end{pmatrix}. \quad (17)$$

This 4×4 matrix, apart from the overall constant factor $1/(mc)^2$ and the minus sign due to the signature convention, precisely equals the corresponding non-relativistic tensor discussed by Unruh [3] and Visser [7].

3 Acoustic metric

We now show that equation (15) with (14) can be put in the form of the d'Alembertian equation of motion for a massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry described by the acoustic metric tensor $G_{\mu\nu}$. We proceed in a way analogous to [7].

We first introduce the acoustic metric tensor $G_{\mu\nu}$ and its inverse $G^{\mu\nu}$ such that

$$f^{\mu\nu} = \sqrt{-G} G^{\mu\nu}, \quad (18)$$

where

$$G = \det(f^{\mu\nu}). \quad (19)$$

These equations imply

$$\det(G_{\mu\nu}) = \det(G^{\mu\nu})^{-1} = G. \quad (20)$$

It follows from equations (14) and (18) that the metric $G_{\mu\nu}$ must be of the form

$$G_{\mu\nu} = k[g_{\mu\nu} - (1 - c_s^2)u_\mu u_\nu], \quad (21)$$

where k is a conformal factor which will be fixed later. Indeed, one may easily check that using this expression and (18) one obtains

$$G^{\mu\gamma}G_{\gamma\nu} = k \frac{n\sqrt{-g}}{w\sqrt{-G}} \delta_\nu^\mu. \quad (22)$$

To calculate the determinant of $G_{\mu\nu}$, it is convenient to use comoving coordinates [9]. In the general comoving coordinate system, the four-velocity vector is given by

$$u^\mu = \frac{\delta_0^\mu}{\sqrt{g_{00}}}; \quad u_\mu = \frac{g_{\mu 0}}{\sqrt{g_{00}}} \quad (23)$$

and hence

$$G = k^4 \det \left(g_{\mu\nu} - (1 - c_s^2) \frac{g_{\mu 0} g_{\nu 0}}{g_{00}} \right). \quad (24)$$

This expression may be simplified using standard algebraic properties of a determinant. We find

$$G = k^4 c_s^2 g. \quad (25)$$

This equation combined with (22) fixes the factor k as

$$k = \frac{n}{wc_s}. \quad (26)$$

Thus, we have shown that the acoustic wave equation may be written in the form

$$\frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} G^{\mu\nu}) \partial_\nu \varphi = 0, \quad (27)$$

with the acoustic metric tensor given by (21) and its inverse by

$$G^{\mu\nu} = \frac{1}{k} \left[g^{\mu\nu} - (1 - \frac{1}{c_s^2}) u^\mu u^\nu \right]. \quad (28)$$

4 Ergosphere and acoustic horizon

In analogy to the Kerr black hole we may now define the *ergo-region* as a region in space-time where the stationary Killing vector becomes space-like. The magnitude $||\xi||$ of ξ^μ may be calculated in terms of the three velocity v defined in appendix A. Using equation (21) (and (84)-(86) from appendix A) we find

$$||\xi||^2 \equiv G_{\mu\nu}\xi^\mu\xi^\nu = k\xi^\mu\xi_\mu[1 - (1 - c_s^2)\gamma^2] = k\xi^\mu\xi_\mu\gamma^2(c_s^2 - v^2). \quad (29)$$

This becomes negative when the magnitude of the flow velocity v exceeds the speed of sound c_s . Thus, as in non-relativistic acoustics [7], any region of supersonic flow is an ergo region. The boundary of the ergo-region is a hypersurface Σ_{c_s} defined by the equation

$$v^2 - c_s^2 = 0. \quad (30)$$

Since the stationary Killing field ξ^μ becomes null on Σ_{c_s} , this hypersurface is called *stationary limit surface* [10]. The hypersurface Σ_{c_s} may, in general, be quite complicated, in particular if the fluid equation of state, and hence the speed of sound, is space-time dependent. For simplicity, in the following discussion we assume $c_s = \text{const}$ throughout the fluid.

To define acoustic horizon we use the concept of wave velocity of a hypersurface [11]. Let $\{\Sigma_a\}$ denote a set of hypersurfaces defined by

$$v^2 - a^2 = 0, \quad (31)$$

where a is a constant, $0 \leq a < c$. The stationary limit surface Σ_{c_s} is contained in $\{\Sigma_a\}$. Equation (31) together with the equation

$$x_0 = \text{const} \quad (32)$$

determines a two-dimensional surface σ_a . The normal to each Σ_a is given by the vector field

$$n_\mu \propto \partial_\mu v^2. \quad (33)$$

Now we define

$$v_\perp = -\frac{n^\mu u_\mu}{\sqrt{(n^\mu u_\mu)^2 - n^\mu n_\mu}} \quad (34)$$

as the *wave velocity* of two-dimensional surfaces σ_a as measured by the observer u , whose world-line is such that the four-velocity u_μ evaluated at a point of Σ_a is tangent to the world-line of u . Clearly, $v_\perp^2 \leq c^2$ if and only if Σ_a is time-like or null, i.e. if and only if $n^\mu n_\mu \leq 0$. If Σ_a is time-like, n_μ may be normalized as $n^\mu n_\mu = -1$. In this case, we decompose u^μ as

$$u^\mu = v_\perp \gamma_\perp n^\mu + \gamma_\perp L^\mu, \quad (35)$$

where

$$\gamma_\perp = \frac{1}{\sqrt{1 - v_\perp^2}}, \quad (36)$$

and

$$L^\mu = \frac{1}{\gamma_\perp} (g^{\mu\nu} + n^\mu n^\nu) u_\nu \quad (37)$$

is a time-like unit vector which represents a displacement in Σ_a along the projection of u^μ . We can regard (35) as a decomposition of the fluid flow into normal and tangential components with respect to Σ_a . The tangential three-velocity v_\parallel^i may be found by decomposing the vector L^μ in a way similar to (85) from appendix A:

$$L^\mu = \gamma_\parallel t^\mu + (g^{\mu\nu} - t^\mu t^\nu) L_\nu, \quad (38)$$

where

$$\gamma_\parallel = t_\mu L^\mu = \frac{1}{\sqrt{1 - v_\parallel^2}}, \quad (39)$$

and

$$v_\parallel^i = \frac{1}{\gamma_\parallel} L^i; \quad i = 1, 2, 3. \quad (40)$$

For steady flow, $t^\mu n_\mu = 0$, and hence equation (35), together with (86) and (39), yields

$$\gamma = \gamma_\parallel \gamma_\perp. \quad (41)$$

Now we define the *acoustic horizon* as a hypersurface \mathcal{H} defined by the equation

$$\frac{(n^\mu u_\mu)^2}{(n^\mu u_\mu)^2 - n^\mu n_\mu} - c_s^2 = 0, \quad (42)$$

i.e. a hypersurface the wave velocity of which equals the speed of sound at every point. The acoustic horizon \mathcal{H} and the stationary limit surface Σ_{c_s} in general do not coincide. Equation (41) states that \mathcal{H} is inside the ergo-region and overlaps with Σ_{c_s} at those points where $L^\mu = t^\mu$, i.e. at the points where the three velocity v^i is perpendicular to the twosurface σ_{c_s} .

Let us illustrate our general considerations by an example. Consider a two-dimensional axisymmetric flow in a stationary axisymmetric space-time. Physically, this may be a model for the equatorial slice of a rotating star, or for a draining flow in an axially symmetric bathtub. In the coordinates t, ϕ, r , the general form of a stationary axisymmetric metric is given by [12]

$$ds^2 = -V(r)(dt - W(r)d\phi)^2 - V(r)^{-1}r^2d\phi^2 - \Omega(r)^2dr^2, \quad (43)$$

where V, W and Ω are arbitrary functions of r approaching $V, \Omega \rightarrow 1$ and $W \rightarrow 0$ as r tends to infinity. We further assume that these functions are positive and non-singular in the region occupied by the fluid. According to equation (93), the fluid velocity takes the form

$$u^\mu = \gamma \left(V^{-1/2} - W v_\phi; v_\phi, v_r \right), \quad (44)$$

where

$$v^2 = V^{-1}r^2v_\phi^2 + \Omega^2v_r^2, \quad (45)$$

with v_ϕ and v_r being the r -dependent azimuthal and radial components, respectively. The surfaces σ_a defined by equation (31) with (32) are concentric circles with the centre at the

origin. The unit vectors n^μ and L^μ appearing in the decomposition of u^μ in (35) may now be easily constructed using (33) and (37), respectively. We find

$$n^\mu = \Omega^{-1} (0; 0, 1) , \quad (46)$$

$$L^\mu = \gamma_\parallel \left(V^{-1/2} - W v_\phi; v_\phi, 0 \right) , \quad (47)$$

where

$$v_\parallel^2 = V^{-1} r^2 v_\phi^2. \quad (48)$$

From equation (34), the wave velocity v_\perp of hypersurfaces σ_a is

$$v_\perp = \gamma_\parallel \Omega v_r . \quad (49)$$

The equation of continuity and the condition that the flow be irrotational constrain the radial dependence of v_\perp and v_\parallel . From equation (1) we find

$$n \gamma_\perp v_\perp r = c_1 \quad (50)$$

and similarly, as a consequence of equation (82), we obtain

$$w \gamma_\perp \gamma_\parallel \left(V v_\parallel - \frac{V^{1/2} W}{r} \right) = c_2, \quad (51)$$

where c_1 and c_2 are arbitrary constants. Of course, these expressions are valid as long as v_\perp^2 and v_\parallel^2 do not exceed the speed of light squared $c^2 = 1$. If we further assume that the particle number n and the speed of sound c_s are constant throughout the fluid, equation (50) yields an explicit expression for the horizon radius

$$r_H = \frac{\sqrt{1 - c_s^2} |c_1|}{c_s n}. \quad (52)$$

The radius of the stationary limit surface can be found by solving the equation

$$r = \frac{\sqrt{1 - c_s^2}}{c_s} \left[\frac{c_1^2}{n^2} + \left(\frac{c_2}{w} + \frac{V(r)^{1/2} W(r)}{\sqrt{1 - c_s^2}} \right)^2 V(r) \right]^{1/2}. \quad (53)$$

Evidently, the acoustic horizon coincides with the stationary limit surface, i.e. the boundary of the ergosphere, if both the azimuthal velocity v_ϕ and the off-diagonal metric term $W(r)$ vanish. It is remarkable that the horizon radius r_H does not depend on metric, whereas the radius of the ergosphere does.

5 Acoustic surface gravity and Hawking temperature

Having established the definition of the acoustic horizon, we proceed to calculate surface gravity at the horizon, restricting attention to stationary acoustic geometries. If the flow is

steady, and the metric $g_{\mu\nu}$ is stationary, there exists a system of coordinates in which the components of $G_{\mu\nu}$ are time independent and the acoustic geometry is said to be *stationary*.

We shall define the surface gravity in terms of the Killing field χ^μ that is null on the horizon. The surface gravity κ may then be calculated using [12]

$$G^{\mu\nu}\partial_\nu||\chi||^2 = -2\kappa\chi^\mu \quad (54)$$

evaluated on the horizon, where

$$||\chi||^2 = G_{\alpha\beta}\chi^\alpha\chi^\beta. \quad (55)$$

Since $||\chi||^2 = 0$ on the horizon, the derivative $\partial_\mu||\chi||^2$ is normal to the horizon, so that on the horizon we have

$$G^{\mu\nu}n_\nu\frac{\partial}{\partial n}||\chi||^2 = 2\kappa\chi^\mu, \quad (56)$$

where $\partial/\partial n \equiv n^\mu\partial_\mu$ denotes the normal derivative on the horizon. The definition of surface gravity by equation (54) is conformally invariant [13]. Therefore, in the calculations that follow we drop the conformal factor k in $G_{\mu\nu}$.

First, consider the simplest case when the stationary limit surface Σ_{c_s} and the acoustic horizon \mathcal{H} coincide. Then, χ^μ is equal to the stationary Killing field ξ^μ which, according to equation (29), is null on the horizon. A straightforward calculation gives

$$G^{\mu\nu}n_\nu = n^\mu - \frac{1}{v\gamma}u^\mu = -\frac{1}{v}\frac{\xi^\mu}{\sqrt{\xi^\nu\xi_\nu}} \quad (57)$$

on the horizon. Equation (56) applied to ξ^μ together with (57) and (29) yields

$$\kappa = \frac{\sqrt{\xi^\nu\xi_\nu}}{1 - c_s^2}\frac{\partial}{\partial n}(v - c_s), \quad (58)$$

where it is understood that the derivative and the quantity $\xi^\nu\xi_\nu$ are to be taken at the horizon. The corresponding Hawking temperature $T_H = \kappa/(2\pi)$ represents the temperature as measured by an observer near infinity. The locally measured temperature follows the Tolman law, i.e.

$$T = \frac{\kappa}{2\pi\sqrt{\xi^\nu\xi_\nu}}, \quad (59)$$

so that the local temperature approaches

$$T \rightarrow \frac{1}{1 - c_s^2}\frac{\partial}{\partial n}(v - c_s) \quad (60)$$

as the acoustic horizon \mathcal{H} is approached. Thus, in this limit, equation (59) with (58) corresponds to the flat space-time Unruh effect. This behaviour is analogous to the Schwarzschild black hole [14] except that our local temperature does not diverge as one approaches \mathcal{H} .

Next, we consider the situation when \mathcal{H} and Σ_{c_s} do not coincide. In this case, the calculation of surface gravity may, in general, be quite non-trivial. As in non-relativistic acoustics [7], there is no particular reason to expect the acoustic horizon to be in general a Killing horizon. However, the matter may greatly simplify if we assume a certain symmetry, such that an analogy with the familiar stationary axisymmetric black hole may be drawn.

We consider the acoustic geometry that satisfies the following assumptions:

1. The acoustic metric is stationary.
2. The flow is symmetric under the transformation that generates a displacement on the horizon along the projection of the flow velocity.
3. The metric $g_{\mu\nu}$ is invariant to the above transformation.

The last two statements are equivalent to saying that a displacement along the projection of the flow velocity on the horizon is an isometry. The axisymmetric geometry considered in section 4 is an example that fulfils the above requirements.

Suppose we have identified the acoustic horizon as a hypersurface \mathcal{H} defined by equation (42). Consider the vector field L^μ defined by (37). Its magnitude is given by

$$||L||^2 \equiv G_{\mu\nu} L^\mu L^\nu = [1 - (1 - c_s^2)\gamma_\perp^2] = \gamma_\perp^2(c_s^2 - v_\perp^2) \quad (61)$$

Hence, the vector L^μ is null on the horizon. We now construct a Killing vector χ^μ in the form

$$\chi^\mu = \xi^\mu + \omega\psi^\mu, \quad (62)$$

where the Killing vector ψ^μ is the generator of the isometry group of displacements on the horizon, and the constant ω is chosen such that χ^μ becomes parallel to L^μ and hence null on the horizon. The quantity ω is analogous to the horizon angular velocity of the axisymmetric black hole [12].

We use a special coordinate system in which the Killing vector ψ has the components

$$\psi^\mu = \delta_\psi^\mu, \quad (63)$$

where the subscript ψ denotes the spatial coordinate x^ψ associated to the above-mentioned isometry. The metric components $g_{\mu\nu}$ and the flow velocity field are, by assumptions 2 and 3, independent of x^ψ . In this coordinate system, the vector L^μ takes the form

$$L^\mu = \gamma_\parallel \left(\frac{1}{\sqrt{g_{00}}} - \frac{g_{0\psi}v_\psi}{g_{00}}; v_\psi, 0, 0 \right). \quad (64)$$

Requiring this vector to be parallel to χ^μ on the horizon, we set

$$\chi^\mu = \sqrt{\chi^\nu \chi_\nu} L^\mu, \quad (65)$$

with

$$\sqrt{\chi^\nu \chi_\nu} = \gamma_\parallel^{-1} \left(\frac{1}{\sqrt{g_{00}}} - \frac{g_{0\psi}v_\psi}{g_{00}} \right)^{-1}. \quad (66)$$

This equation combined with (62) determines the “angular velocity” of the horizon

$$\omega = \left(\frac{1}{\sqrt{g_{00}}} - \frac{g_{0\psi}v_\psi}{g_{00}} \right)^{-1} v_\psi. \quad (67)$$

The calculation of the surface gravity is now straightforward. The magnitude of χ^μ in the vicinity of the horizon is given from equation (61) by

$$||\chi||^2 = \gamma_\perp^2(c_s^2 - v_\perp^2)\chi^\nu \chi_\nu. \quad (68)$$

Using the definition of L_μ (37) and equation (65), we find

$$G^{\mu\nu}n_\nu = n^\mu - \frac{1}{v_\perp \gamma_\perp} u^\mu = -\frac{1}{v_\perp} \frac{\chi^\mu}{\sqrt{\chi^\nu \chi_\nu}} \quad (69)$$

on the horizon. Equation (56), together with (68) and (69), yields

$$\kappa = \frac{\sqrt{\chi^\nu \chi_\nu}}{1 - c_s^2} \frac{\partial}{\partial n} (v_\perp - c_s), \quad (70)$$

where the values of the derivative and the quantity $\chi^\nu \chi_\nu$ are to be taken at the horizon. The non-relativistic limit of this equation in flat space-time coincides with the expression for surface gravity derived by Visser [7].

6 Concluding Remarks

We have studied the relativistic acoustics in curved space-time in terms of an acoustic metric tensor. The acoustic metric tensor involves a non-trivial explicit space-time dependence. Apart from that, all the features of the non-relativistic acoustic geometry in flat space-time [3, 7] have their relativistic counterparts. We have shown that the propagation of sound in a general-relativistic perfect fluid with an irrotational flow may be described by a scalar d'Alembert equation in a curved acoustic geometry. The acoustic metric tensor is a matrix that depends on the fluid equation of state and involves two tensors: the background space-time metric tensor $g_{\mu\nu}$ and the product of the fluid four-velocities $u_\mu u_\nu$. We have then shown that if in a non-homogeneous flow there exists a supersonic region called *ergo-region*, then an acoustic event horizon forms. We have discussed an example of axisymmetric geometry where the acoustic horizon does not coincide with the boundary of the ergo-region. We have calculated the surface gravity κ using a conformally invariant definition [13] which involves a properly defined Killing vector that is null on the horizon. An acoustic horizon emits Hawking radiation of thermal phonons [1] at the temperature $T_H = \kappa/(2\pi)$ as measured by an observer near infinity.

We believe that the effects we have discussed may be of physical interest in all those phenomena that involve relativistic fluids under extreme conditions. This may be the case in astrophysics, early cosmology and ultrarelativistic heavy-ion collisions.

A Basic Fluid Dynamics

Consider a perfect gravitating relativistic fluid. We denote by u_μ , p , ρ , n and s the velocity, pressure, energy density, particle number density and entropy density of the fluid. The energy-momentum tensor of a perfect fluid is given by

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \quad (71)$$

where $g_{\mu\nu}$ is the metric tensor with the Lorentzian signature $(+ - - -)$. Hence, in this convention, we have

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = 1. \quad (72)$$

Our starting point is the continuity equation

$$(nu^\mu)_{;\mu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} nu^\mu) = 0, \quad (73)$$

and the energy-momentum conservation

$$T^{\mu\nu}_{;\nu} = 0. \quad (74)$$

This equation applied to (71) yields the relativistic Euler's equation [8]

$$(p + \rho)u^\nu u_{\mu;\nu} + \partial_\mu p + u_\mu u^\nu \partial_\nu p = 0. \quad (75)$$

Euler's equation may be further simplified if one restricts consideration to an isentropic flow. A flow is said to be *isentropic* when the specific entropy s/n is constant, i.e. when

$$\partial_\mu \left(\frac{s}{n} \right) = 0. \quad (76)$$

A flow may in general have a non-vanishing vorticity $\omega_{\mu\nu}$ defined as

$$\omega_{\mu\nu} = h^\rho_\mu h^\sigma_\nu u_{[\rho;\sigma]}, \quad (77)$$

where

$$h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu \quad (78)$$

is the projection operator which projects an arbitrary vector in space-time into its component in the subspace orthogonal to u^μ . A flow with vanishing vorticity, i. e. when

$$\omega_{\mu\nu} = 0, \quad (79)$$

is said to be *irrotational*. In the following we assume that the flow is isentropic and irrotational.

As a consequence of equation (76) and the thermodynamic identity

$$dw = Td\left(\frac{s}{n}\right) + \frac{1}{n}dp, \quad (80)$$

with $w = (p + \rho)/n$ being the specific enthalpy, equation (75) simplifies to

$$u^\nu (wu_\mu)_{;\nu} - \partial_\mu w = 0. \quad (81)$$

Furthermore, for an isentropic irrotational flow, equation (79) implies [11]

$$(wu_\mu)_{;\nu} - (wu_\nu)_{;\mu} = 0. \quad (82)$$

In this case, we may introduce a scalar function ϕ such that

$$wu_\mu = -\partial_\mu \phi, \quad (83)$$

where the minus sign is chosen for convenience. It may be easily seen that the quantity wu^μ in the form (83) satisfies equation (81). Solutions of this form are the relativistic analogue of potential flow in non-relativistic fluid dynamics [8].

It is convenient to parameterize the components of the fluid four-velocity in terms of three-velocity components. To do this, we use the projection operator $g_{\mu\nu} - t_\mu t_\nu$, which projects a vector into the subspace orthogonal to the time translation Killing vector $\xi^\mu = (1; \vec{0})$, where t_μ is the unit vector

$$t^\mu = \frac{\xi^\mu}{\sqrt{\xi^\nu \xi_\nu}} = \frac{\delta_0^\mu}{\sqrt{g_{00}}}; \quad t_\mu = \frac{\xi_\mu}{\sqrt{\xi^\nu \xi_\nu}} = \frac{g_{\mu 0}}{\sqrt{g_{00}}}. \quad (84)$$

We split up the vector u_μ in two parts: one parallel and the other orthogonal to t_μ :

$$u_\mu = \gamma t_\mu + (g_{\mu\nu} - t_\mu t_\nu) u^\nu, \quad (85)$$

where

$$\gamma = t^\mu u_\mu. \quad (86)$$

From (85) with (84) we find

$$u_0 = \gamma \sqrt{g_{00}}. \quad (87)$$

Equations (85), (86) and the normalization of u_μ imply

$$\gamma_{ij} u^i u^j = \gamma^2 - 1, \quad (88)$$

where γ_{ij} is the induced three-dimensional spatial metric:

$$\gamma_{ij} = \frac{g_{0i} g_{0j}}{g_{00}} - g_{ij}; \quad i, j = 1, 2, 3. \quad (89)$$

Now, it is natural to introduce the three-velocity v^i , so that

$$u^i = \gamma v^i, \quad (90)$$

with its covariant components v_i and the magnitude squared v^2 given by

$$v_i = \gamma_{ij} v^j, \quad v^2 = v^i v_i. \quad (91)$$

It follows from (88), (90) and (91) that

$$\gamma^2 = \frac{1}{1 - v^2}. \quad (92)$$

Since u^μ and t^μ are time-like unit vectors, a consequence of (85) is that $\gamma \geq 1$ and hence $0 \leq v^2 < 1$.

Equations (84) - (91) may now be used to calculate other covariant and contravariant components of the fluid four-velocity. We find

$$\begin{aligned} u^\mu &= \gamma \left(\frac{1}{\sqrt{g_{00}}} - \frac{g_{0j} v^j}{g_{00}}; v^i \right), \\ u_\mu &= \gamma \left(\sqrt{g_{00}}; \frac{g_{0i}}{\sqrt{g_{00}}} - v_i \right). \end{aligned} \quad (93)$$

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